Shock Profiles for the Asymmetric Simple Exclusion Process in One Dimension

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Abstract

The asymmetric simple exclusion process (ASEP) on a one-dimensional lattice is a system of particles which jump at rates p and $1-p$ (here $p > 1/2$) to adjacent empty sites on their right and left respectively. The system is described on suitable macroscopic spatial and temporal scales by the inviscid Burgers' equation; the latter has shock solutions with a discontinuous jump from left density ρ_- to right density ρ_+, $\rho_- < \rho_+$, which travel with velocity $(2p-1)(1-\rho_+-\rho_-)$. In the microscopic system we may track the shock position by introducing a second class particle, which is attracted to and travels with the shock. In this paper we obtain the time invariant measure for this shock solution in the ASEP, as seen from such a particle. The mean density at lattice site n, measured from this particle, approaches $\rho_\pm$ at an exponential rate as $n \to \pm\infty$, with a characteristic length which becomes independent of p when $p/(1-p) > \sqrt{\rho_+(1-\rho_-)/\rho_-(1-\rho_+)}$. For a special value of the asymmetry, given by $p/(1-p) = \rho_+(1-\rho_-)/\rho_-(1-\rho_+)$, the measure is Bernoulli, with density ρ_- on the left and ρ_+ on the right. In the weakly asymmetric limit, $2p-1 \to 0$, the microscopic width of the shock diverges as $(2p-1)^{-1}$. The stationary measure is then essentially a superposition of Bernoulli measures, corresponding to a convolution of a density profile described by the viscous Burgers equation with a well-defined distribution for the location of the second class particle.

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1. Introduction

The aim of this and our previous work [1, 2], with S. Janowsky, is the determination of the underlying microscopic structure of a fluid in regions in which it has shocks on the macroscopic scale. We consider situations in which the system evolves macroscopically according to some deterministic autonomous equations, such as the Euler, Navier-Stokes, or Burgers' equations [3, 4, 5]. When the solutions of these equations are smooth, we can assume that on the microscopic level the system is (essentially) in a local equilibrium state, determined by the local macroscopic parameters obtained from the solutions. What is less clear, however, and is of particular interest, is what happens when the macroscopic evolution is not smooth, as in the occurrence of shocks; these are described by regions of very large gradient in solutions of the Navier-Stokes or viscous Burgers' equations and by discontinuities in solutions of their zero viscosity limits, the Euler or inviscid Burgers' equations. It is first of all not clear in what sense the macroscopic equations are to be interpreted in such regions, since their derivation (heuristic or rigorous) is based on the assumption of slow variation in the system's properties on the microscopic scale. Beyond that, these equations do not describe the structure of the shocks on the microscopic scale. This is the problem we wish to address here. In particular, do the statistical properties of the atoms or molecules change abruptly on the interparticle distance scale? Or do these properties change significantly only over much larger distances? (For a discussion of shock structure in gases based on the "mesoscopic" description provided by the Boltzmann equations, see [7].)

One model for which these questions have been answered, at least partially, is the asymmetric simple exclusion process (ASEP) [6, 8] on the one-dimensional lattice $\mathbb{Z}$. In this model particles attempt at random times (distributed as independent Poisson processes of unit density at each site) to jump to an adjacent site, choosing the site on their right with some fixed probability p and that on their left with probability $q = 1-p$; the attempt succeeds if the target site is not already occupied. For any value of p the set of extremal translation invariant stationary states is the set of Bernoulli measures ν_ρ $(0 \le \rho \le 1)$ [8]; in the state ν_ρ, each lattice site is occupied independently with probability ρ and there is therefore a current $(2p-1)\rho(1-\rho)$. The dynamics satisfies detailed balance with respect to ν_ρ if and only if the transitions are symmetric $(p = 1/2)$.

The macroscopic mass density $u(y,t)$, which is an appropriately scaled continuum limit of the particle density in the microscopic model, is described [9]–[12] by the inviscid Burgers' equation

$$u_t + (2p-1)[u(1-u)]_y = 0. \tag{1.1}$$

It is well known [5] that solutions of (1.1) can exhibit shocks. In the simplest example, $u(y,t) = \rho_+$ for $y > y_0(t)$ and $u(y,t) = \rho_-$ for $y < y_0(t)$, where $0 \le \rho_- < \rho_+ \le 1$; the shock position $y_0(t)$ moves with the constant velocity $(2p-1)(1-\rho_+-\rho_-)$, as is easily determined from conservation of mass. It is then natural to ask what this discontinuity of the field u means at the microscopic scale.

At the microscopic level, a configuration of the ASEP at time t is fully specified by the occupation numbers $\tau_i(t)$, taking values 0 and 1, of all the sites on the lattice. To describe the profile of the shock, we must first locate the position of the shock in each configuration. This is a nontrivial problem, since we have to distinguish between the variations of density due to intrinsic fluctuations and those due to the presence of the shock. Fortunately there is, for this system, a simple way of defining the position of the shock at the microscopic level: the introduction of a single second class particle into the system. (Alternative and more general methods will be discusssed in [13].) The second class particle acts like a hole with respect to the original, first class particles and like a particle with respect to holes, and thus its presence does not affect at all the dynamics of the first class particles. Specifically, during any infinitesimal time interval dt, exchanges occur on each bond as follows:

$$\left.\begin{array}{ccc} 1\,0 & \rightarrow & 0\,1 \\ 2\,0 & \rightarrow & 0\,2 \\ 1\,2 & \rightarrow & 2\,1 \end{array}\right\} \text{ with probability } p\,dt\,,$$
$$\left.\begin{array}{ccc} 0\,1 & \rightarrow & 1\,0 \\ 0\,2 & \rightarrow & 2\,0 \\ 2\,1 & \rightarrow & 1\,2 \end{array}\right\} \text{ with probability } q\,dt\,, \tag{1.2}$$

where a 0, 1 or 2 on a site means that this site is occupied by a hole, a regular (first class) particle or the second class particle.

It can be shown [15] (see also [16, 17], and [1] for a heuristic argument) that the velocity of the second class particle in a uniform environment of first class particles at density ρ is $(p-q)(1-2\rho)$, so that far to the left of the shock the second class particle moves at a velocity $(p-q)(1-2\rho_-)$, faster than the shock velocity $(p-q)(1-\rho_- - \rho_+)$, whereas far to the right of the shock the second class particle has a velocity $(p-q)(1-2\rho_+)$, slower than that of the shock. Consequently, the second class particle is attracted to the shock and can serve as a marker for its position. It has in fact been proved [15] that the second class particle moves with velocity $(p-q)(1-\rho_- - \rho_+)$ and [18, 15] (see also [14], [19]–[22]) that there is an invariant measure for the system viewed from this second class particle, in which the asymptotic densities are $\rho_\pm$.

In the present paper we describe exactly this invariant measure. Our approach is an extension of a matrix method which has been used in a number of situations [24]–[35], in which the weight of each configuration is written as the matrix element of a matrix product. As in [1], there are three possibilities for each matrix in the product, D, A and E, depending on whether the corresponding site is occupied by a first class particle, occupied by a second class particle, or is empty. We show in Section 2 and Appendix A that when the matrices D, A and E satisfy certain algebraic rules, these matrix products furnish weights for the invariant measure we seek. In contrast to most previous cases in which this method was used, however, here we write probabilities of events in the invariant measure, using matrix products, *directly in the infinite system.* This approach was used in [34] to recover the results of [1].

The invariant measures are parametrized by the pair of densities ρ_- and ρ_+. There is a special value of the asymmetry parameter p, given by $p/(1-p) = \rho_+(1-\rho_-)/\rho_-(1-\rho_+)$, at which the measures are Bernoulli, with density ρ_- to the left of the second class particle and ρ_+ to its right. This is described in Section 3, where we also show that the algebraic rules given in Section 2 imply certain symmetry properties of the invariant measure.

The full expression for the shock profile, as seen from the second class particle, is described in Section 4, and is derived in Section 5 by constructing an explicit representation of the algebra of Section 2.

We study in Section 6 the asymptotic behavior of this profile at large distances from the second class particle, and show that it decays at an exponential rate to ρ_- or ρ_+. The decay length is a function of the parameters p and $\rho_\pm$ which, however, becomes independent of the asymmetry p for $p/(1-p) > \sqrt{\rho_+(1-\rho_-)/\rho_-(1-\rho_+)}$.

Finally, in Section 7 we show that in the weak asymmetry regime, where $p = (1+\epsilon)/2$ with $0 < \epsilon \ll 1$, the shock profile as seen from the second class particle can be understood as the convolution of the hyperbolic tangent profile predicted by the viscous Burgers equation,

$$u_t + [u(1-u)]_y = (1/2)u_{yy}, \tag{1.3}$$

which is known [36, 37] to describe this regime (on a longer time scale than that on which (1.1) holds), with the density of the position of the second class particle in this tanh profile, given by the derivative of the profile.

Throughout the paper, the shock will be characterized by the two asymptotic densities ρ_- to the left of the second class particle and ρ_+ to the right of the second class particle, which satisfy $0 \le \rho_- < \rho_+ \le 1$, and by the hopping rates p to the right and $q = 1-p$ to the left, as in (1.2). We will often express our results in terms of ρ_+, ρ_-, p and q through the parameters

$$x = \frac{q}{p}\ , \quad a = \rho_+(1-\rho_-)\ , \quad b = \rho_-(1-\rho_+)\ . \tag{1.4}$$

2. The algebra for the partially asymmetric shock measure

In analogy with [1, 24] we write the probability of a configuration specified by the occupation numbers τ_i ($\tau_i = 0,1$) of m consecutive sites to the left of the second class particle (which is located at the origin) and n consecutive sites to its right as a matrix element of the form

$$\langle w| \left\{ \prod_{i=-m}^{-1} [\tau_i D + (1-\tau_i)E] \right\} A \left\{ \prod_{j=1}^{n} [\tau_j D + (1-\tau_j)E] \right\} |v\rangle. \tag{2.1}$$

Here, as in [1], a first class particle is represented by a matrix D, a hole by a matrix E, and the second class particle by a matrix A. For example, the probability of finding the configuration 1 0 1 to the left and 0 1 1 0 0 to the right of the second class particle, that is, of the configuration

$$1\ 0\ 1\ 2\ 0\ 1\ 1\ 0\ 0\ , \tag{2.2}$$

is given by

$$\langle w|\ D\ E\ D\ A\ E\ D^2\ E^2\ |v\rangle. \tag{2.3}$$

In Appendix A we show, by an extension of the proof which was given in [1], that if the matrices D, E and A and the vectors $\langle w|$ and $|v\rangle$ satisfy certain algebraic conditions then the weights (2.1) are nonnegative and define an invariant measure for the ASEP, as seen from a single second class particle, with dynamics specified by (1.2). The algebra is

$$\begin{aligned} pDE - qED &= (p-q)[(1-\rho_-)(1-\rho_+)D + \rho_-\rho_+E], & (2.4)\\ pAE - qEA &= (p-q)(1-\rho_-)(1-\rho_+)A, & (2.5)\\ pDA - qAD &= (p-q)\rho_+\rho_-A, & (2.6)\\ (D+E)|v\rangle &= |v\rangle, & (2.7)\\ \langle w|(D+E) &= \langle w|, & (2.8)\\ \langle w|A|v\rangle &= 1. & (2.9) \end{aligned}$$

At this stage the numbers ρ_+ and ρ_- which appear in (2.4)–(2.9) are arbitrary parameters satisfying $0 \le \rho_- < \rho_+ \le 1$. However, we will show in Section 6 that they are in fact the two asymptotic densities which are reached as one moves away from the second class particle.

As a consequence of (2.1) and of (2.4)–(2.9) the microscopic profile, defined here as the average occupation $\langle \tau_n \rangle$ at position n, is given to the right of the second class particle by

$$\langle \tau_n \rangle = \langle w|\ A\ (D+E)^{n-1}\ D\ |v\rangle, \quad n \ge 1, \tag{2.10}$$

and to the left of the second class particle by

$$\langle \tau_{-n} \rangle = \langle w | D \, (D+E)^{n-1} A | v \rangle, \quad n \le -1. \tag{2.11}$$

In contrast to the approach to the shock problem taken in [1], expressions like (2.10) and (2.11) are valid directly for the infinite system; thus here we avoid completely the difficulty of taking the infinite volume limit.

By an argument similar to that of Sandow [25] we may verify that (2.4)–(2.9) suffice to determine any matrix element containing precisely one factor of A. To do so we will show that any matrix element of a product of $n+1$ operators can be reduced to a sum of matrix elements of products of n operators. Consider, for example, a matrix element of the form $\langle w|DO_n|v\rangle$, where O_n is a product of n operators which contains a single A. Then

$$\begin{aligned} \langle w|DO_n|v\rangle &= x^{k+1}\langle w|O_n D|v\rangle + \text{l.o.t.} \\ &= -x^{k+1}\langle w|O_n E|v\rangle + \text{l.o.t.} \\ &= -x^{n+1}\langle w|EO_n|v\rangle + \text{l.o.t.} \\ &= x^{n+1}\langle w|DO_n|v\rangle + \text{l.o.t.}\ . \end{aligned} \tag{2.12}$$

Here l.o.t. (lower order terms) denotes matrix elements of products of n matrices, $x = q/p$ as in (1.4), and k is the number of factors of E in O_n. To obtain (2.12) we have used first (2.4) and (2.6), then (2.7), then (2.4) and (2.5), and finally (2.8). Since $x < 1$, equation (2.12) can be solved for $\langle w|DO_n|v\rangle$. This reduction permits the calculation of matrix elements of arbitrary length.

It is important to realize, however, that the algebraic rules do not allow us to calculate matrix elements of products of operators which contain either no operator A or more than one such operator. For example, $\langle w|DE|v\rangle$ or even $\langle w|v\rangle$ cannot be calculated from these rules alone. Of course, if one has a representation of the matrices D, A, and E and of the vectors $\langle w|$ and $|v\rangle$ which satisfies (2.4-2.9) then one can in principle calculate these other matrix elements. The values thus obtained depend on the representation and do not appear relevant for the problem we consider here.

When n is small the reduction described above is easy to carry out and one can calculate directly the average occupation numbers of the sites closest to the second class

particle, as well as other simple correlation functions. Thus for example

$$
\begin{aligned}
\langle \tau_1 \rangle &= \frac{\rho_+ + \rho_- \;-\; (1-x)\rho_+\rho_-}{1+x}, \\
\langle \tau_{-1} \rangle &= \frac{x(\rho_+ + \rho_-) \;+\; (1-x)\rho_+\rho_-}{1+x}, \\
\langle \tau_1\tau_2 \rangle &= \frac{\rho_+^2 + \rho_+\rho_- + \rho_-^2}{1+x+x^2} + \frac{(-1-x+2x^2)\rho_+\rho_-(\rho_+ + \rho_-) + x(1-x)^2(\rho_+\rho_-)^2}{(1+x+x^2)\;(1+x)}, \\
\langle \tau_{-1}\tau_1 \rangle &= \frac{x(\rho_+^2 + \rho_+\rho_- + \rho_-^2) + (1-x)^2\rho_+\rho_-(\rho_+ + \rho_- - \rho_+\rho_-)}{1+x+x^2}, \\
\langle \tau_{-2}\tau_{-1} \rangle &= \frac{x^2(\rho_+^2 + \rho_+\rho_- + \rho_-^2)}{1+x+x^2} + \frac{(2x-x^2-x^3)\rho_+\rho_-(\rho_+ + \rho_-) + (1-x)^2(\rho_+\rho_-)^2}{(1+x+x^2)\;(1+x)}.
\end{aligned}
\tag{2.13}
$$

In principle one could compute arbitrary correlation functions in this way, but the calculation quickly becomes impractical as the number of sites involved increases. We note that, in contrast with the totally asymmetric case $x = 0$, discussed in [1], there are in general correlations between the left and the right of the second class particle. The expressions in (2.13) are symmetric polynomial functions of ρ_+ and ρ_-. The symmetry follows from (2.4)–(2.6), but we could not derive it by an elementary argument from the dynamics.

Finally, we observe that if one has a representation of D, E, $|v\rangle$, and $\langle w|$ which satisfies (2.4), (2.7), and (2.8) and for which $\langle w|DE - ED|v\rangle$ is finite and nonzero, then one may satisfy (2.5)–(2.6) by defining

$$A = c\;(DE - ED), \tag{2.14}$$

where c is a constant fixed by the condition (2.9). The representation constructed in Section 5 is obtained in this way. We must emphasize, however, that there are representations of interest in which (2.14) is not satisfied. One such representation is used in Section 3 below to study the system at a special value x^* of x; note also that if one wished to use the algebra (2.4)–(2.9) when $x = 1$ then (2.4) would imply that D and E commute, so that (2.14) would be inconsistent with the normalization (2.9). More generally, (2.7) and (2.8) imply that $\langle w|DE|v\rangle = \langle w|ED|v\rangle$, so that $\langle w|DE - ED|v\rangle$ can be nonzero only if $\langle w|DE|v\rangle$ and $\langle w|ED|v\rangle$ are infinite. Thus finite dimensional representations of the algebra cannot satisfy (2.14); the representation constructed in Section 5 is, as expected from this remark, infinite dimensional.

3. Elementary consequences of the algebra

We present here two simple consequences of the representation of the invariant measure described in Section 2.

First we note that there is a special value x^* of the ratio $x = q/p$ for which the measure becomes a Bernoulli measure with density ρ_+ to the right of the second class particle and density ρ_- to the left of the second class particle:

$$x^* = \frac{\rho_-(1-\rho_+)}{\rho_+(1-\rho_-)} = \frac{b}{a} \; . \tag{3.1}$$

This can be seen by verifying that when $x = x^*$ the following formulas define a two-dimensional representation the algebra (2.4)–(2.9):

$$D = \begin{pmatrix} \rho_+ & 0 \\ 0 & \rho_- \end{pmatrix} , \quad E = \begin{pmatrix} 1-\rho_+ & 0 \\ 0 & 1-\rho_- \end{pmatrix} , \quad A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} ;$$
$$\langle w| = (0,1) \; , \quad |v\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \; . \tag{3.2}$$

It is also easy to check directly that this Bernoulli measure is stationary. When $x = x^*$ the profile on both sides of the second class particle is flat ($\langle\tau_n\rangle = \rho_+$ and $\langle\tau_{-n}\rangle = \rho_-$ for all $n \geq 1$) and the occupation numbers at all sites are independent:

$$\langle\tau_{-m}\ldots\tau_{-1}\tau_1\ldots\tau_n\rangle = \langle\tau_{-m}\rangle\ldots\langle\tau_{-1}\rangle\langle\tau_1\rangle\ldots\langle\tau_n\rangle = \rho_-^m\rho_+^n. \tag{3.3}$$

The value (3.1) of x^* plays the role of a disorder line in equilibrium statistical mechanics models [39, 40].

Second, we obtain a number of identities satisfied by the invariant measure for all x, ρ_-, and ρ_+. These will be derived under the assumption that A has the special form (2.14), but this is not a restriction on their validity, since we know that there exists at least one representation, constructed in Section 5, for which this is true. The simplest of these identities is

$$\begin{aligned} \langle\tau_n\rangle - \langle\tau_{n+1}\rangle &= \langle w|A(D+E)^{n-1}(DE-ED)|v\rangle \\ &= \langle w|(DE-ED)(D+E)^{n-1}A|v\rangle \\ &= \langle\tau_{-n-1}\rangle - \langle\tau_{-n}\rangle \; , \end{aligned} \tag{3.4}$$

which implies that the shock profile has the symmetry

$$\langle\tau_n\rangle + \langle\tau_{-n}\rangle = \langle\tau_{n+1}\rangle + \langle\tau_{-n-1}\rangle \; . \tag{3.5}$$

From the known results (2.13) for the first sites it follows that the common value is

$$\langle\tau_n\rangle + \langle\tau_{-n}\rangle = \rho_+ + \rho_- \ . \tag{3.6}$$

Note that the right hand side of (3.6) is consistent with the asymptotic values $\langle\tau_n\rangle \to \rho_+$ and $\langle\tau_{-n}\rangle \to \rho_-$ as $n \to \infty$.

One can obtain in the same manner identities involving higher correlations. For example, from (2.14) one has

$$\begin{aligned}
\langle w|A(D+E)^{n'}(DE-ED)(D+E)^{m'}D|v\rangle &= \langle w|(DE-ED)(D+E)^{n'}A(D+E)^{m'}D|v\rangle,\\
\langle w|A(D+E)^{n'}D(D+E)^{m'}(DE-ED)|v\rangle &= \langle w|(DE-ED)(D+E)^{n'}D(D+E)^{m'}A|v\rangle,\\
\langle w|D(D+E)^{n'}A(D+E)^{m'}(DE-ED)|v\rangle &= \langle w|D(D+E)^{n'}(DE-ED)(D+E)^{m'}A|v\rangle,
\end{aligned} \tag{3.7}$$

and by choosing $n' = n-1$ and $m' = m-1$ one obtains symmetry relations involving pair correlation functions

$$\begin{aligned}
\langle\tau_n\tau_{n+m+1}\rangle - \langle\tau_{n+1}\tau_{n+m+1}\rangle &= \langle\tau_{-n-1}\tau_m\rangle - \langle\tau_{-n}\tau_m\rangle,\\
\langle\tau_n\tau_{n+m}\rangle - \langle\tau_n\tau_{n+m+1}\rangle &= \langle\tau_{-n-m-1}\tau_{-m}\rangle - \langle\tau_{-n-m}\tau_{-m}\rangle,\\
\langle\tau_{-n}\tau_m\rangle - \langle\tau_{-n}\tau_{m+1}\rangle &= \langle\tau_{-n-m-1}\tau_{-m-1}\rangle - \langle\tau_{-n-m-1}\tau_{-m}\rangle,
\end{aligned} \tag{3.8}$$

which, through linear combinations, lead to the fact that $\langle\tau_n\tau_{n+m}\rangle + \langle\tau_{-n}\tau_m\rangle + \langle\tau_{-n-m}\tau_{-m}\rangle$ does not depend on n or m and therefore is given from (2.13) by

$$\langle\tau_n\tau_{n+m}\rangle + \langle\tau_{-n}\tau_m\rangle + \langle\tau_{-n-m}\tau_{-m}\rangle = \rho_+^2 + \rho_+\rho_- + \rho_-^2 \tag{3.9}$$

Note that this common value is again consistent with the asymptotic densities $\rho_\pm$ and the independence of the occupation numbers $\tau_{\pm n}$, $\tau_{\pm m}$ as $n, m \to \infty$ (see Remark 6.1).

4. The expression for the density profile

Here we present an expression for the profile $\langle\tau_n\rangle$, based on an explicit representation of the algebra (2.4)–(2.9) which will be constructed in Section 5. The formula for the profile to the right of the second class particle is

$$\langle\tau_n\rangle = \rho_+ + \sum_{k=-\infty}^{\infty} P_{n-1}(k) F_k \ , \qquad \text{for } n \geq 1, \tag{4.1}$$

where the $P_n(k)$ are defined by the recursion

$$\begin{aligned} P_0(k) &= \delta_{k,0}, & (4.2)\\ P_{n+1}(k) &= aP_n(k-1) + (1-a-b)P_n(k) + bP_n(k+1), & (4.3) \end{aligned}$$

and F_k is given by

$$F_k = \frac{1}{a-b}(a^2 f_{k+2} - a(a+b)f_{k+1} + b(a+b)f_{k-1} - b^2 f_{k-2}), \tag{4.4}$$

with

$$f_k = k\,\frac{x^k}{1-x^k}, \qquad \text{for } k \neq 0. \tag{4.5}$$

f_0 can be defined arbitrarily because its contribution to (4.1) cancels out: indeed, it is easy to check that (4.2) and (4.3) imply that $a^k P_n(-k) = b^k P_n(k)$ for all k, so that the coefficient of f_0 in (4.1) is always zero. (We will introduce below a particular value of f_0 which is convenient in intermediate computations.) The profile to the left of the second class particle—$\langle\tau_n\rangle$ for $n \leq -1$—can easily be obtained from the symmetry (3.6).

More complicated correlation functions have similar expressions, which can be derived in the same way or, in some cases, directly from (4.1) and the algebra. For example, from (2.4)–(2.9), one can show that

$$(1-x)D^2|v\rangle = [D(D+E) - x(D+E)D - (1-x)(1-\rho_+-\rho_-)D - (1-x)\rho_+\rho_-]|v\rangle, \tag{4.6}$$

and this implies that

$$(1-x)\langle\tau_n\tau_{n+1}\rangle = \langle\tau_n\rangle - x\langle\tau_{n+1}\rangle - (1-x)(1-\rho_+-\rho_-)\langle\tau_n\rangle - (1-x)\rho_+\rho_-, \tag{4.7}$$

which gives, using (4.1),

$$\langle\tau_n\tau_{n+1}\rangle = \rho_+^2 + \sum_{k=-\infty}^{\infty} P_{n-1}(k)\ G_k \ , \tag{4.8}$$

where

$$G_k = (\rho_+ + \rho_-)F_k + \frac{x}{1-x}[(a+b)F_k - aF_{k+1} - bF_{k-1}] \ . \tag{4.9}$$

Remark 4.1: The quantity $P_n(k)$ given by (4.2) and (4.3) can be interpreted as the probability of finding a biased random walker on site k at time n, given that it was at the origin at time 0. We did not find a simple physical interpretation for the presence of this random walk in the expression (4.1) for the profile but we believe that understanding the origin of this biased random walk would give a better insight in the whole problem of the description of a shock as seen from a second class particle.

5. Representation of the algebra

We now describe the explicit representation of the algebra (2.4)–(2.9) used to derive the formulas of Section 4 for the profile and to study the weak asymmetric limit in Section 7. The operators D, E, and A will act on an infinite dimensional vector space with basis $\{\,|n\rangle \mid n = 0, \pm1, \pm2, \ldots\}$. Let us define the left shift operator L, the two diagonal operators S and T, and a vector $|v\rangle$ and dual vector $\langle w|$ by

$$\begin{aligned} L\,|n\rangle &= |n-1\rangle \ , && (5.1)\\ T\,|n\rangle &= x^n(1+x^n)^{-1}|n\rangle \ , && (5.2)\\ S\,|n\rangle &= \Big[x^n(1+x^n)^{-1} - x^{n-1}(1+x^{n-1})^{-1}\Big]|n\rangle \ , && (5.3)\\ |v\rangle &= \sum_{n=-\infty}^{\infty}|n\rangle \ , \qquad \langle w| = \sum_{n=-\infty}^{\infty}\langle n| \ . && (5.4) \end{aligned}$$

These are easily seen to satisfy the following relations:

$$pLT - qTL = (p-q)TLT, \qquad pTL^{-1} - qL^{-1}T = (p-q)TL^{-1}T; \tag{5.5}$$

$$(p-q)T(aL - bL^{-1})T = paLT - qaTL - pbTL^{-1} + qbL^{-1}T; \tag{5.6}$$

$$TL - LT = -LS, \qquad L^{-1}T - TL^{-1} = -SL^{-1}; \tag{5.7}$$

$$L|v\rangle = |v\rangle, \qquad \langle w|L = \langle w|. \tag{5.8}$$

Next we define the operators of our representation:

$$\begin{aligned} D &= \rho_-\rho_+ + aL - (\sqrt{a}L - \sqrt{b})T(\sqrt{a} + \sqrt{b}L^{-1}), && (5.9)\\ E &= (1-\rho_-)(1-\rho_+) + bL^{-1} + (\sqrt{a}L - \sqrt{b})T(\sqrt{a} + \sqrt{b}L^{-1}), && (5.10)\\ A &= -(a-b)^{-2}(DE - ED). && (5.11) \end{aligned}$$

Then the formula (2.4) for $pDE - qED$ follows immediately from (5.6), and as noted in Section 2 the relations (2.5)–(2.6) for $pAE - qEA$ and $pDA - qAD$ are automatically satisfied by (5.11). Moreover

$$D + E = aL + (1 - a - b)I + bL^{-1}, \tag{5.12}$$

so that $(D + E)|v\rangle = |v\rangle$ and $\langle w|(D + E) = \langle w|$. Finally, (5.7) implies that

$$A = -(a - b)^{-2}(\sqrt{a}L - \sqrt{b})(aLS - bSL^{-1})(\sqrt{a} + \sqrt{b}L^{-1}), \tag{5.13}$$

and from $\langle w|S|v\rangle = -1$ it follows that $\langle w|A|v\rangle = 1$.

Note that, as expected from the remark following (2.14), the matrix elements $\langle w|v\rangle$, $\langle w|DE|v\rangle$ and $\langle w|ED|v\rangle$ are all infinite. In this representation the presence of a factor A in a matrix product—or more specifically, that of S (see (5.13)), a diagonal matrix whose diagonal elements S_{jj} decrease exponentially fast to 0 as $j \to \pm\infty$—renders the $\langle w| \cdot |v\rangle$ matrix element of that product finite.

We now turn to the evaluation of $\langle \tau_n \rangle$. First we observe that, for $k > 0$,

$$\begin{aligned}
\sum_{j=-N}^{N} \frac{x^{j-k}}{1 + x^{j-k}} \frac{x^j}{1 + x^j} &= \sum_{j=-N}^{N} \frac{1}{1 - x^k} \left(\frac{x^j}{1 + x^j} - x^k \frac{x^{j-k}}{1 + x^{j-k}} \right) \\
&= \sum_{j=-N}^{N} \frac{x^j}{1 + x^j} + \frac{x^k}{1 - x^k} \left(\sum_{j=N-k+1}^{N} \frac{x^j}{1 + x^j} - \sum_{j=-N-k}^{-N-1} \frac{x^j}{1 + x^j} \right).
\end{aligned} \tag{5.14}$$

From (5.14) we have, for $k > 0$,

$$\begin{aligned}
\langle w|SL^kT|v\rangle &= \lim_{N\to\infty} \sum_{j=-N}^{N} \left(\frac{x^{j-k}}{1 + x^{j-k}} - \frac{x^{j-1-k}}{1 + x^{j-1-k}} \right) \frac{x^j}{1 + x^j} \\
&= \lim_{N\to\infty} \left[\frac{x^k}{1 - x^k} \left(\sum_{j=N-k+1}^{N} \frac{x^j}{1 + x^j} - \sum_{j=-N-k}^{-N-1} \frac{x^j}{1 + x^j} \right) \right. \\
&\qquad \left. - \frac{x^{k+1}}{1 - x^{k+1}} \left(\sum_{j=N-k}^{N} \frac{x^j}{1 + x^j} - \sum_{j=-N-k-1}^{-N-1} \frac{x^j}{1 + x^j} \right) \right] \\
&= f_{k+1} - f_k,
\end{aligned} \tag{5.15}$$

with $f_k = kx^k/(1 - x^k)$ as in (4.5). From the easily verified identity

$$\langle w|SL^jT|v\rangle + \langle w|SL^{-j-1}T|v\rangle = -1, \tag{5.16}$$

it then follows that (5.15) also holds for $k < -1$, with f_k again given by (4.5). Now let us define $f_0 = f_1 - \langle w|ST|v\rangle$; then (5.15) holds for $k = 0$ and also $k = -1$ (using (5.16) again), and thus for all k. It is in fact not necessary to evaluate f_0, by the remark following (4.5).

Using (5.9), (5.13), and (5.15) we have, for all k,

$$\begin{aligned} \langle w|AL^kD|v\rangle &= \rho_-\rho_+ + a + (a-b)^{-1}\langle w|(aLS - bSL^{-1})L^k(aL - bL^{-1})T|v\rangle \\ &= \rho_+ + (a-b)^{-1}\langle w|S(a^2L - ab - abL^{-1} + b^2L^{-2})L^kT|v\rangle \\ &= \rho_+ + (a-b)^{-1}\Big(a^2 f_{k+2} - a(a+b)f_{k+1} + b(a+b)f_{k-1} - b^2 f_{k-2}\Big). \end{aligned} \tag{5.17}$$

The formula (5.12) for $D+E$ yields by induction

$$(D+E)^n = \sum_k P_n(k)L^k \tag{5.18}$$

for $n \geq 0$, where the $P_n(k)$ are defined by (4.2) and (4.3). Thus for $n \geq 1$,

$$\begin{aligned} \langle\tau_n\rangle &= \langle w|A(D+E)^{n-1}D|v\rangle \\ &= \sum_{k=-\infty}^{\infty} P_{n-1}(k)\langle w|AL^kD|v\rangle \\ &= \rho_+ + \frac{1}{a-b}\sum_{k=-\infty}^{\infty} P_{n-1}(k)\Big(a^2 f_{k+2} - a(a+b)f_{k+1} + b(a+b)f_{k-1} - b^2 f_{k-2}\Big), \end{aligned} \tag{5.19}$$

and this completes the derivation of the formula (4.1) for $\langle\tau_n\rangle$.

From the expression (4.1) it is possible to evaluate various asymptotics of the profile. The two main limits that one might wish to consider are the limit $n \to \infty$ describing the tail of the profile, discussed in the following section (where we also comment briefly on the $\rho_+ \searrow \rho_-$ limit), and the limit of a weak asymmetry, $x \to 1$, discussed in Section 7.

6. Spatial asymptotics of the density profile

We now discuss the behavior of the profile $\langle\tau_n\rangle$ at large n, basing our discussion on (4.1). Because $a > b$, the biased random walker governed by (4.3) goes to $+\infty$ as $n \to \infty$, with distribution $P_n(k)$ concentrated around $k \simeq n(a-b)$. Moreover, F_k approaches 0 as k approaches ∞, decaying exponentially (as kx^k), so that the sum in (4.1) vanishes in the $n \to \infty$ limit and thus the asymptotic density on the right is ρ_+, i.e.,

$$\langle\tau_n\rangle \to \rho_+ \qquad \text{as } n \to \infty. \tag{6.1}$$

From the symmetry (3.6), the asymptotic density at the left is ρ_-.

To obtain the approach of $\langle \tau_n \rangle$ to its $n \to \infty$ limit, we need to estimate how the sum on the right hand side of (4.1) vanishes. As observed above, F_k decays exponentially as $k \to \infty$, and it is easy to check that it approaches the constant value $-(a-b)$ as $k \to -\infty$. Because of this behavior there are two cases to consider: the sum is dominated either by large values of k or by values of k close to 0, depending on the value of x.

Case I: $x > (b/a)^{1/2}$

We first explore the consequences of assuming that the sum in (4.1) is dominated by large values of k. If this is so, one can use the approximation

$$f_k \simeq k x^k, \tag{6.2}$$

and then the identity

$$\sum_{k=-\infty}^{\infty} x^k P_n(k) = \left(1 - a - b + ax + \frac{b}{x}\right)^n . \tag{6.3}$$

This leads to

$$\begin{aligned}\langle \tau_n \rangle - \rho_+ \simeq (n-1)&\frac{(ax - b/x)^2\,(ax - a - b + b/x)}{a-b}\left(1 - a - b + ax + \frac{b}{x}\right)^{n-2} \\ &+ \frac{2a^2x^2 - (a+b)ax - (a+b)b/x + 2b^2/x^2}{a-b}\left(1 - a - b + ax + \frac{b}{x}\right)^{n-1} .\end{aligned} \tag{6.4}$$

The values of k which dominate the sum (6.3) are $k \simeq n(ax - b/x)/(1 - a - b + ax + b/x)$, so that the assumption that large values of k dominate the sum in (4.1) would be inconsistent if $x^2 < b/a$. Conversely, when $x^2 > b/a$, (6.4) gives the asymptotic behavior of $\langle \tau_n \rangle$.

Case II: $x < (b/a)^{1/2}$

Since F_k is constant for $k \to -\infty$ and $P_n(k)$ increases with k for $k < 0$, it is clear that the sum over k must be dominated by values of k near $k = 0$ (i.e. which do not scale with n). It is convenient to use the following expression for the solution $P_n(k)$ of (4.3):

$$P_n(k) = \sum_{m=0}^{n} \binom{n}{m} (1-a-b)^{n-m} \binom{m}{\frac{m+k}{2}} b^{\frac{m-k}{2}} a^{\frac{m+k}{2}} . \tag{6.5}$$

For large m and for k of order 1, one can write

$$\begin{aligned}\binom{m}{\frac{m+k}{2}} &\simeq 2^m \sqrt{\frac{2}{\pi m}}\left(1 - \frac{k^2}{2m}\right), \quad \text{for } m+k \text{ even,} \\ \binom{m}{\frac{m+k}{2}} &= 0, \quad \text{for } m+k \text{ odd.}\end{aligned} \tag{6.6}$$

Then

$$\begin{aligned}\langle\tau_n\rangle - \rho_+ &\simeq \frac{1}{2}\sqrt{\frac{2}{\pi}}\sum_{m=0}^{n-1}\binom{n-1}{m}(1-a-b)^{n-m-1}\,(ab)^{m/2}\,\frac{2^m}{m^{1/2}}\sum_{k=-\infty}^{\infty}\left(\frac{a}{b}\right)^{k/2}\left(1-\frac{k^2}{2m}\right)F_k \\ &\simeq \frac{1}{2}\sqrt{\frac{2}{\pi}}\sum_{m=0}^{n-1}\binom{n-1}{m}(1-a-b)^{n-m-1}2^m\,(ab)^{m/2}\,\frac{Z}{2m^{3/2}}. \end{aligned} \tag{6.7}$$

Here we have used the fact that $\sum_{k=-\infty}^{\infty}(a/b)^{k/2}F_k = 0$ (see (4.4) and (4.5)) and set

$$Z = -\sum_{k=-\infty}^{\infty} k^2\left(\frac{a}{b}\right)^{k/2} F_k = -\frac{4\sqrt{ab}\,(\sqrt{a}-\sqrt{b})^2}{a-b}\sum_{k=-\infty}^{\infty}\frac{k^2x^k}{1-x^k}\,\left(\frac{a}{b}\right)^{k/2}. \tag{6.8}$$

The extra factor of $1/2$ in (6.7) comes from the fact that we have replaced a sum in which m and k have the same parity by a sum over all m and k. Finally, using the fact that $\sum_m\binom{n}{m}x^{n-m}y^m m^{-\alpha} \simeq (x+y)^{n+\alpha}(ny)^{-\alpha}$, since this sum is dominated by values of m near $ny/(x+y)$, (6.7) becomes

$$\langle\tau_n\rangle - \rho_+ \simeq \frac{Z}{8\sqrt{\pi}\,(ab)^{3/4}}\,\frac{1}{n^{3/2}}\,[1-(\sqrt{a}-\sqrt{b})^2]^{n+\frac{1}{2}}. \tag{6.9}$$

By pairing the $\pm k$ terms in (6.8) we see that if $x < x^* = b/a$ then $Z > 0$ and asymptotically $\langle\tau_n\rangle > \rho_+$, while if $x > x^*$, asymptotically $\langle\tau_n\rangle < \rho_+$. Numerical computation indicates that $\langle\tau_n\rangle - \rho_+$ has the same sign for all n and is in fact monotonically decreasing for $x < x^*$ and monotonically increasing for $x > x^*$. Note also that Z vanishes for $x = x^*$, as expected from the special nature of the measure at this value of x (see Section 3).

When $x = 0$, (6.8) becomes

$$Z = \frac{4ab}{(\sqrt{a}-\sqrt{b})^2}, \tag{6.10}$$

leading to

$$\langle\tau_n\rangle - \rho_+ \simeq \frac{(ab)^{1/4}}{2\sqrt{\pi}(\sqrt{a}-\sqrt{b})^2}\,\frac{1}{n^{3/2}}\,[1-(\sqrt{a}-\sqrt{b})^2]^{n+\frac{1}{2}}. \tag{6.11}$$

This agrees with (6.26) of [1] (and with (7.6) of that paper after the correction of a misprint: the exponent $2i-1$ should be $2i+1$).

By a modification of the above calculation one may also find the asymptotics when $\rho_+ = \rho_-$. The result is independent of x and is that given in [1]: $\langle\tau_n\rangle \simeq \rho + \sqrt{\rho(1-\rho)/\pi n}$ when $n \gg 1$, where $\rho = \rho_+ = \rho_-$.

The various asymptotics for the profile derived here are summarized in Figure 1. Figure 2 shows the profile for some typical parameter values.

Remark 6.1 Using the explicit representation of the algebra developed in Section 5 it can be verified that the measure is asymptotically Bernoulli as $n \to \pm\infty$, i.e., that if Γ is any operator product containing j factors of D and k factors of E then

$$\lim_{n\to\infty} \langle w|A(D+E)^n\Gamma|v\rangle = \rho_+^j(1-\rho_+)^k, \tag{6.12}$$

with a similar limit at $-\infty$. The approach is exponentially fast, with the same decay rate as for the profile, that is, $(1-a-b+ax+b/x)^n$ when $x > (b/a)^{1/2}$ and $[1-(\sqrt{a}-\sqrt{b})^2)]^n$ when $x < (b/a)^{1/2}$. More generally, if Δ is a product of D's, E's and precisely one A, then $\langle w|\Delta(D+E)^n\Gamma|v\rangle \to \langle w|\Delta|v\rangle\rho_+^j(1-\rho_+)^k$ as $n \to \infty$, with the same rate of approach (and again a similar limit at $-\infty$).

7. The weakly asymmetric limit

In this section we study the limiting behavior of the model as the asymmetry $\epsilon = p-q$ becomes small, with $\rho_\pm$ fixed. In this case the profile $\langle\tau_n\rangle$ becomes very broad and as $\epsilon \to 0$ depends only on the scaled variable $y = n\epsilon$. To verify this, note that for ϵ small and k large it follows from

$$x = q/p = 1 - 2\epsilon + O(\epsilon^2) \tag{7.1}$$

that

$$f_{k+p} = (k+p)\frac{x^{k+p}}{1-x^{k+p}} \simeq k\frac{x^k}{1-x^k} + p\frac{x^k}{1-x^k} - 2pk\epsilon\frac{x^k}{(1-x^k)^2}, \tag{7.2}$$

so that

$$F_k \simeq (a-b)\left[\frac{x^k}{1-x^k} - 2k\epsilon\frac{x^k}{(1-x^k)^2}\right]. \tag{7.3}$$

Now for large n the probability $P_n(k)$ is concentrated around $k = n(a-b)$; if we introduce the variable $y = n\epsilon$, use the approximation (7.3), and set $k = (a-b)y/\epsilon$ we obtain

$$\begin{aligned}\langle\tau_n\rangle &\simeq \rho_+ + \lambda\left[\frac{2e^{-4\lambda y}}{1-e^{-4\lambda y}} - \lambda y\frac{8e^{-4\lambda y}}{(1-e^{-4\lambda y})^2}\right] \\ &\simeq \frac{\rho_+ + \rho_-}{2} + \lambda\left[\coth 2\lambda y - \frac{2\lambda y}{\sinh^2 2\lambda y}\right],\end{aligned} \tag{7.4}$$

where $\lambda = (a-b)/2 = (\rho_+ - \rho_-)/2$. It is easy to see that the terms neglected in (7.4) are $O(\epsilon)$ when y is of order 1. Note that the asymptotic behavior of (7.4) agrees with the $x \to 1$ limit of (6.4).

This scaling form for the profile can be understood in terms of the viscous Burgers equation for a hydrodynamic density variable $u(y,s)$:

$$u_s + [u(1-u)]_y = (1/2)u_{yy}. \tag{7.5}$$

It is shown in [36], [37] (see also [38]) that the weakly asymmetric macroscopic limit of the ASEP, in scaled variables $y = n\epsilon$ as above and s, related to the microscopic time t by $s = \epsilon^2 t$, is indeed (7.5). For example, if the initial state of the particle system with parameter ϵ is a product measure with density $u_0(\epsilon n)$ at site n, then the observed density $u(y,s)$ at position $\epsilon^{-1}y$ and time $\epsilon^{-2}s$ is the solution of (7.5) with initial condition $u(y,0) = u_0(y)$. Equation (7.5) is diffusive and does not exhibit discontinuous shocks; instead of the jump in density from ρ_- to ρ_+ we now have a smooth profile $u(y,s) = \rho(y - vs)$, traveling with velocity $v = 1 - \rho_+ - \rho_-$, with

$$\rho(y) = (\rho_- + \rho_+)/2 + \lambda \tanh 2\lambda y, \tag{7.6}$$

and $\lambda = (\rho_+ - \rho_-)/2$. Thus the shock width on the microscopic scale diverges in this limit as ϵ^{-1}.

We can now interpret (7.4), after putting $y = \epsilon n$, as the shock profile seen from a second class particle which does not have a fixed location relative to the tanh profile; instead, its position is distributed in such a way that it sees all densities between ρ_- and ρ_+ with equal probability. Then the probability of finding the second class particle between y and $y + dy$ is $d\rho(y) = Q(y)\,dy$, where

$$Q(y) = \frac{1}{\rho_+ - \rho_-}\rho'(y) = \lambda \cosh^{-2} 2\lambda y, \tag{7.7}$$

and the limit of the one particle density function will be

$$\lim_{\epsilon\to 0}\langle \tau_{\lfloor \epsilon^{-1} y\rfloor}\rangle = \int_{-\infty}^{\infty} \rho(y+z)Q(z)\,dz = \frac{\rho_+ + \rho_-}{2} + \lambda\left[\coth\lambda y - \frac{2\lambda y}{\sinh^2 2\lambda y}\right], \tag{7.8}$$

in agreement with (7.4); here $\lfloor z\rfloor$ denotes the greatest integer k satisfying $k \le z$.

Remark 7.1: In [18] the shock is viewed, not from the position of a second class particle, but from a location defined in a rather less direct manner. With that definition the profile seen in the weakly asymmetric limit is $\rho(y)$, not the convolution (7.8).

There are various possible direct justifications for the density (7.7). A heuristic argument may be based on a comparison of two systems: in one of these the second class particle is treated as a first class particle and in the other as a hole. In the latter case

the shock position, however determined, is effectively translated a macroscopic distance $\epsilon/(\rho_+ - \rho_-)$ relative to its position in the former. Thus the density of the second class particle is $(\rho(y + \epsilon/(\rho_+ - \rho_-)) - \rho(y))/\epsilon \simeq \rho'(y)/(\rho_+ - \rho_-)$. A rigorous version of this argument is given in [15].

The above picture is confirmed if we compute other aspects of the $\epsilon \to 0$ limiting behavior of the entire invariant measure. Consider first the distribution of occupation numbers at a finite number of sites with both microscopic and macroscopic spacing. To give a concrete example we consider

$$\langle w|A(D+E)^{n_1}DED(D+E)^{n_2-n_1}EEDD|v\rangle \tag{7.9}$$

where $n_1 = \lfloor y_1/\epsilon \rfloor$ and $n_2 = \lfloor y_2/\epsilon \rfloor$ for some macroscopic positions y_1, y_2 with $0 < y_1 < y_2$; this is the probability that of three specific sites at y_1 the first and third are occupied and the second empty, and that the first two of four sites at y_2 are empty and the second two occupied. Then

$$\begin{aligned}\lim_{\epsilon \to 0} &\langle w|A(D+E)^{n_1}DED(D+E)^{n_2-n_1}EEDD|v\rangle \\ &= \int_{-\infty}^{\infty} \rho(y_1+z)^2(1-\rho(y_1+z))\rho(y_2+z)^2(1-\rho(y_2+z))^2\, Q(z)\, dz.\end{aligned} \tag{7.10}$$

We may think of the right hand side of (7.10) as arising from a distribution of particles in which sites at macroscopic positions y are occupied with probability $\rho(y)$ and all occupation numbers are independent; this distribution is viewed from a random position z (the position of the second class particle) distributed according to $Q(z)\,dz$. Equation (7.10)—in fact, its generalization to an arbitrary number of sites—will be proved in Appendix B; in particular, this furnishes an alternate derivation of the special case (7.4) of one site.

We can also show that, in a certain sense, the typical configuration in the $\epsilon \to 0$ limit has on the macroscopic scale the shape of the tanh profile (7.6). To do so we introduce a coarse graining on an intermediate scale $\epsilon^{-\beta}$, where $0 < \beta < 1$: for each macroscopic position y we define a random variable $B_\epsilon(y)$, the empirical density in a block of size $\epsilon^{-\beta}$ at y, by

$$B_\epsilon(y) = \epsilon^\beta \sum_{k=1}^{\lfloor \epsilon^{-\beta} \rfloor} \tau_{\lfloor y/\epsilon \rfloor + k}. \tag{7.11}$$

Now consider, for very small ϵ, a configuration of the system which after coarse graining looks like the tanh profile (7.6) on the macroscopic scale. The second class particle will be located at some macroscopic position z relative to the center of this tanh profile and hence will see locally a density $\rho(z)$; thus $B_\epsilon(0) \simeq \rho(z)$ and $B_\epsilon(0)$ determines the position

z by $z \simeq \rho^{-1}(B_\epsilon(0))$. At position y relative to the second class particle the density will be $\rho(z+y)$, and we are led to expect a strong correlation between the empirical densities at positions 0 and y relative to the second class particle, given by

$$B_\epsilon(y) \simeq \rho(y + \rho^{-1}(B_\epsilon(0))). \tag{7.12}$$

We show in Appendix B that (7.12) holds with arbitrary accuracy, and with probability arbitrarily close to one, for sufficiently small ϵ, that is, that for any $\eta > 0$,

$$\lim_{\epsilon\to 0} \text{Prob}\left[B_\epsilon(y) - \rho(y + \rho^{-1}(B_\epsilon(0))) > \eta\right] = 0 \tag{7.13}$$

(convergence in probability). Equation (7.13) essentially says that, with probability one, the coarse grained densities in the $\epsilon \to 0$ limit lie on some translate of the profile $\rho(x)$.

8. Concluding remarks

We have found a family of stationary measures for the ASEP as seen from a second class particle, parametrized by two numbers ρ_- and ρ_+, with $0 \le \rho_- < \rho_+ \le 1$, which correspond to densities of the asymptotic Bernoulli measures at $\pm\infty$. The results, which hold for all $p > 1/2$, generalize those of [1], in which such measures were obtained for the fully asymmetric case $p = 1$. Our derivation, in contrast to that of previous works, is carried out directly in the infinite system.

For a certain value of the asymmetry, $x = x^*$, we found a two dimensional representation of the algebra (see (3.1) and (3.2)). In fact, it can be shown that a $2r$ dimensional representation exists when $x^r = x^*$, $r = 1, 2, \ldots$. Finite dimensional representations of other ASEP algebras have been found in [29, 43].

In the weak asymmetry limit $p \to 1/2$, the measure can be understood in terms of the solutions of the macroscopic viscous Burgers equation. Several recent works have studied the gap of the evolution operator or generator [41, 42], as well as the diffusion constant [33], on a ring of size L, in the double limit $p \to 1/2$, $L \to \infty$. A simple scaling form was found for the diffusion constant; it would be nice to see whether a scaling form in either problem could also be derived from a macroscopic equation.

The microscopic shock profile derived in this work contains both intrinsic features of the shock and the fluctuations of the position of the second class particle. One can imagine many other ways of locating the shock [13]; with an alternate definition, the expression for the profile would certainly be different. However, one expects that there exist intrinsic properties of the shock which are independent of the definition of its location. One class of

such properties would be the values of sums of expectations of all translates of quantities whose expectation values vanish in the asymptotic Bernoulli measures with densitites ρ_+ and ρ_-, e.g.,

$$\sum_{i=-\infty}^{\infty} \langle (\tau_i - \rho_-)\tau_{i+n}(\tau_{i+n+m} - \rho_+)\rangle. \tag{8.1}$$

We have evaluated explicitly such sums for the totally asymmetric case $p = 1$, using the invariant measure as seen from the second class particle, and will report on this work in a later publication [13]. From this point of view, identities such as (3.4) reflect the fact that the second class particle can be considered, in the sum, as either a first class particle or a hole.

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Appendix A. Verification that the algebra yields an invariant measure

In this appendix we show that if the operators D, E, and A and the vectors $|v\rangle$ and $\langle w|$ satisfy the algebra (2.4)–(2.9) then the formula (2.1) yields an invariant measure for the ASEP as seen from a second class particle. We begin by writing down the conditions which must be verified. The algebra allows us to calculate directly the probability of finding configurations $\tau_{-m}, \ldots, \tau_{-1}$ to the left, and $\tau_1, \ldots, \tau_n$ to the right, of the second class particle, where the τ_i are thought of as elements of the set $\{0, 1\}$. For convenience in describing the dynamics and consistency with the notation of (1.2) we will write this local configuration as $\tau = (\tau_{-m}, \ldots, \tau_{-1}, \tau_0, \tau_1, \ldots, \tau_n)$; in this notation the symbol τ_0 always takes the value $\tau_0 = 2$ and represents the presence of a second class particle at the origin.

Suppose then that for each such τ we have defined a weight $W(\tau)$. Then these weights define an invariant probability measure for the ASEP if they satisfy (i) normalization conditions:

$$\begin{aligned} W(\tau_0) &= 1, \\ W(\tau_{-m}, \ldots, \tau_{-1}, \tau_0, \tau_1, \ldots, \tau_n) &= \sum_{\sigma=0,1} W(\tau_{-m}, \ldots, \tau_{-1}, \tau_0, \tau_1, \ldots, \tau_n, \sigma) \\ &= \sum_{\sigma=0,1} W(\sigma, \tau_{-m}, \ldots, \tau_{-1}, \tau_0, \tau_1, \ldots, \tau_n); \end{aligned} \tag{A.1}$$

(ii) positivity:

$$W(\tau_{-m},\ldots,\tau_{-1},\tau_0,\tau_1,\ldots,\tau_n)\geq 0\;; \tag{A.2}$$

and (iii) for each configuration τ a condition corresponding to the stationarity of the probability of τ. To write down these stationarity conditions we will use the following notation: if $\tau=(\tau_{-m},\ldots,\tau_n)$ is a system configuration and $-m\leq i\leq n-1$ then $\tau^{i,i+1}$ denotes the configuration obtained from τ by a particle jump across the bond between sites i and $i+1$; if $i,i+1\neq 0$ this is an interchange of τ_i and τ_{i+1}, but if either $i=0$ or $i+1=0$ it is a shift of the configuration relative to the second class particle:

$$\begin{aligned}
\tau_j^{0,1} &= \begin{cases}\tau_1, & \text{if } j=-1,\\ 2, & \text{if } j=0,\\ \tau_{j+1}, & \text{if } -m-1\leq j\leq -2 \text{ or } 1\leq j\leq n-1;\end{cases}\\
\tau_j^{-1,0} &= \begin{cases}\tau_{-1}, & \text{if } j=1,\\ 2, & \text{if } j=0,\\ \tau_{j-1}, & \text{if } -m+1\leq j\leq -1 \text{ or } 2\leq j\leq n+1;\end{cases}\\
\tau_j^{i,i+1} &= \begin{cases}\tau_{i+1}, & \text{if } j=i,\\ \tau_i, & \text{if } j=i+1,\\ \tau_j, & \text{if } j\neq i,i+1,\end{cases} \qquad \text{when } i,i+1\neq 0.
\end{aligned} \tag{A.3}$$

Now during some time interval dt there is, for each pair of adjacent sites $i,i+1$ with $\tau_i\neq\tau_{i+1}$, some probability of exit from the configuration τ by an interchange $\tau\to\tau^{i,i+1}$, and some probability of entrance into τ by an interchange $\tau^{i,i+1}\to\tau$: if $\tau_i\,\tau_{i+1}$ has the form $1\,0$, $1\,2$, or $2\,0$ then the exit probability is $p\,dt$ and the entrance probability $q\,dt$, while if $\tau_i\,\tau_{i+1}$ has the form $0\,1$, $2\,1$, or $0\,2$ then these probabilities are reversed (see (1.2)). Moreover, there are probabilities for exit from and entrance to τ due to exchanges across the ends of τ, between sites $-m$ and $-m-1$ or n and $n+1$; if $n=0$ or $m=0$, these exchanges involve the second class particle. Combining these effects we find the equation

$$\begin{aligned}
0 \;=\; & \sum_{i=-m}^{n-1}{}' [-pW(\tau)+qW(\tau^{i,i+1})] + \sum_{i=-m}^{n-1}{}'' [-qW(\tau)+pW(\tau^{i,i+1})]\\
& +\chi_1(\tau_{-m})[-qW(0\tau)+pW((0\tau)^{-m-1,-m})]\\
& +\chi_0(\tau_{-m})[-pW(1\tau)+qW((1\tau)^{-m-1,-m})]\\
& +\chi_1(\tau_n)[-pW(\tau 0)+qW((\tau 0)^{n,n+1})]\\
& +\chi_0(\tau_n)[-qW(\tau 1)+pW((\tau 1)^{n,n+1})]\;,
\end{aligned} \tag{A.4}$$

where the singly (respectively doubly) primed sum is over indices i for which $\tau_i\tau_{i+1}$ is 10, 12 or 20 (respectively 01, 21 or 02), and the indicator functions χ_1 and χ_0 are defined by

$$\chi_1(\sigma)=\begin{cases}1, & \text{if } \sigma=1 \text{ or } \sigma=2,\\ 0, & \text{if } \sigma=0,\end{cases} \qquad \chi_0(\sigma)=\begin{cases}1, & \text{if } \sigma=0 \text{ or } \sigma=2,\\ 0, & \text{if } \sigma=1.\end{cases} \tag{A.5}$$

Equation (A.4) demands special attention if all the τ_i other than τ_0 are the same, say $\tau_i \equiv 1$, $i \neq 0$. Then exchanges involving the second class particle will not always change the local configuration. For example, if $\tau = (1,1,1,2,1,1)$ then after the exchange $\tau \to \tau^{0,1} = (1,1,1,1,2,1)$ the local configuration is still τ if there was originally a particle on site 3 (the third site to the right of the origin). Thus (A.4), which includes a term $-qW(\tau)$ from this exchange, should be corrected by a term $+qW(\tau 1)$. But (A.4) also contains a term $+qW(\tau^{-1,0})$ from the exchange $\tau^{-1,0} = (1,1,2,1,1,1) \to \tau$; in this case the local configuration was already τ if there was before the exchange a particle at site -3. Thus (A.4) should also be corrected by a term $-qW((1\tau)^{-1,0})$. Since $\tau 1 = (1\tau)^{-1,0}$, these two corrections cancel. Said otherwise, (A.4) counts the transition $(1,1,1,2,1,1,1) \to (1,1,1,1,2,1,1)$, which does not change τ, twice, with opposite signs. The argument is easily seen to be quite general and to apply also to leftward jumps of the second class particle and to the case $\tau_i \equiv 0$, $i \neq 0$. We conclude that (A.4) is correct in all cases.

We will now show that the weights defined by (2.1),

$$W_0(\tau_{-m},\ldots,\tau_{-1},\tau_0,\tau_1,\ldots,\tau_n) = \langle w| \prod_{i=-m}^{-1} [\tau_i D + (1-\tau_i)E] \; A \; \prod_{j=1}^{n} [\tau_j D + (1-\tau_j)E] |v\rangle, \quad \text{(A.6)}$$

satisfy (A.1), (A.2), and (A.4) and hence provide an invariant measure for the ASEP. (A.1) follows immedately from (2.7)–(2.9). We will verify (A.2) by induction on $m+n$, the total number of D and E operators in the product. Since $\langle w|A|v\rangle = 1$ by (2.9) the case $m+n=0$ is immediate. For general $m+n$ it suffices to show that

$$\langle w|E^m A D^n|v\rangle \geq 0, \quad \text{(A.7)}$$

since by repeated use of the relations (2.4)–(2.6) we may express the matrix element of any product with m operators E and n operators D as $x^k\langle w|E^m A D^n|v\rangle$, for some k, plus a linear combination, with positive coefficients, of matrix elements of products with a smaller number of D or E operators.

To verify (A.7) we define new operators $\hat{D}$ and $\hat{E}$ by $D = \rho_+\rho_- + \hat{D}$ and $E = (1-\rho_+)(1-\rho_-) + \hat{E}$ and show, again by induction, that

$$\langle w|\hat{E}^m A \hat{D}^n|v\rangle > 0, \quad \text{(A.8)}$$

from which (A.7) (with strict inequality) follows immediately. The operators $\hat{D}$ and $\hat{E}$ satisfy

$$\hat{D}\hat{E} - x\hat{E}\hat{D} = (1-x)ab, \quad \hat{D}A - xA\hat{D} = 0, \quad A\hat{E} - x\hat{E}A = 0,$$
$$(\hat{D}+\hat{E})|v\rangle = (a+b)|v\rangle, \quad \langle w|(\hat{D}+\hat{E}) = \langle w|(a+b),$$

and hence also $\hat{D}^k\hat{E} - x^k\hat{E}\hat{D}^k = ab(1-x^k)\hat{D}^{k-1}$ and $\hat{D}\hat{E}^k - x^k\hat{E}^k\hat{D} = ab(1-x^k)\hat{E}^{k-1}$ for any $k \geq 1$. Then an argument as in (2.12) leads to the recursions

$$\begin{aligned}
\langle w|\hat{E}^{m+1}A|v\rangle &= \frac{1}{1-x^{m+2}}\Big[(a+b)(1-x^{m+1})\langle w|\hat{E}^m A|v\rangle \\
&\qquad -ab(1-x^m)\langle w|\hat{E}^{m-1}A|v\rangle\Big], \qquad \text{(A.9)}\\
\langle w|\hat{E}^m A\hat{D}^{n+1}|v\rangle &= (a+b)\langle w|\hat{E}^m A\hat{D}^n|v\rangle - x^{n+1}\langle w|\hat{E}^{m+1}A\hat{D}^n|v\rangle \\
&\qquad -ab(1-x^n)\langle w|\hat{E}^m A\hat{D}^{n-1}|v\rangle; \qquad \text{(A.10)}
\end{aligned}$$

similar recursions hold with m and n interchanged, hence $\langle w|\hat{E}^m A\hat{D}^n|v\rangle = \langle w|\hat{E}^n A\hat{D}^m|v\rangle$. From (A.9) it follows by induction on m that

$$\langle w|\hat{E}^m A|v\rangle = \frac{a^{m+1}-b^{m+1}}{a-b}\frac{1-x}{1-x^{m+1}}, \qquad \text{(A.11)}$$

and then from (A.10) and (A.11), by induction on n, that for $n \leq m$,

$$\langle w|\hat{E}^m A\hat{D}^n|v\rangle = \sum_{k=0}^{n}(ab)^k\,\frac{a^{m+n+1-2k}-b^{m+n+1-2k}}{a-b}\,\frac{(1-x)\prod_{j=k+1}^{n}(1-x^j)}{\prod_{j=0}^{n-k}(1-x^{m+1+j})} \qquad \text{(A.12)}$$

(the identity $(a+b)(a^j-b^j) = (a^{j+1}-b^{j+1}) + ab(a^{j-1}-b^{j-1})$ is needed in checking (A.11) and (A.12)). The expression (A.12) is clearly positive, so that (A.8) holds.

Finally, we must verify (A.4). To do so, we partition the matrix product in (A.6) into blocks of consecutive identical matrices—that is, blocks of D's or E's, together with one block consisting of a single A. When the form (A.6) is substituted into the stationarity condition (A.4), each resulting term arises from a possible exchange at some block boundary—between blocks or at the end or beginning of the product—and contains a corresponding factor $\pm(pDE-qED)$, $\pm(pAE-qEA)$, or $\pm(pDA-qAD)$. These factors may be simplified with the fundamental relations (2.4)–(2.6), which we will use in several forms:

$$\begin{aligned}
pDE - qED &= (p-q)(eD+dE) \\
&= (p-q)((d-e)E + e(D+E)) \\
&= (p-q)((e-d)D + d(D+E)), \qquad \text{(A.13)}\\
pAE - qEA &= (p-q)eA, \\
pDA - qAD &= (p-q)dA.
\end{aligned}$$

Here $d = \rho_+\rho_-$ and $e = (1-\rho_-)(1-\rho_+)$.

The net effect of this simplification on each possible block boundary is summarized in the following table:

Boundary	Contribution to (A.4)	Simplified contribution
DE	$\langle w\| \quad \cdots \quad (-pDE+qED) \quad \cdots \quad \|v\rangle$	$(p-q)\langle w\| \cdots (-eD-dE) \cdots \|v\rangle$
ED	$\langle w\| \quad \cdots \quad (+pDE-qED) \quad \cdots \quad \|v\rangle$	$(p-q)\langle w\| \cdots (+eD+dE) \cdots \|v\rangle$
DA	$\langle w\| \quad \cdots \quad (-pDA+qAD) \quad \cdots \quad \|v\rangle$	$(p-q)\langle w\| \quad \cdots \quad (-dA) \quad \cdots \quad \|v\rangle$
AD	$\langle w\| \quad \cdots \quad (+pDA-qAD) \quad \cdots \quad \|v\rangle$	$(p-q)\langle w\| \quad \cdots \quad (+dA) \quad \cdots \quad \|v\rangle$
AE	$\langle w\| \quad \cdots \quad (-pAE+qEA) \quad \cdots \quad \|v\rangle$	$(p-q)\langle w\| \quad \cdots \quad (-eA) \quad \cdots \quad \|v\rangle$
EA	$\langle w\| \quad \cdots \quad (+pAE-qEA) \quad \cdots \quad \|v\rangle$	$(p-q)\langle w\| \quad \cdots \quad (+eA) \quad \cdots \quad \|v\rangle$
$\langle w\|D$	$\langle w\|(+pDE-qED) \quad \cdots \quad \|v\rangle$	$(p-q)\langle w\|((e-d)D+d) \quad \cdots \quad \|v\rangle$
$\langle w\|E$	$\langle w\|(-pDE+qED) \quad \cdots \quad \|v\rangle$	$(p-q)\langle w\|((e-d)E-e) \quad \cdots \quad \|v\rangle$
$\langle w\|A$	$\langle w\|(-pDA+qAD+pAE-qEA)\cdots\|v\rangle$	$(p-q)\langle w\|((e-d)A) \quad \cdots \quad \|v\rangle$
$D\|v\rangle$	$\langle w\| \quad \cdots \quad (-pDE+qED)\|v\rangle$	$(p-q)\langle w\| \quad \cdots \quad ((d-e)D-d)\|v\rangle$
$E\|v\rangle$	$\langle w\| \quad \cdots \quad (+pDE-qED)\|v\rangle$	$(p-q)\langle w\| \quad \cdots \quad ((d-e)E+e)\|v\rangle$
$A\|v\rangle$	$\langle w\|\cdots(+pDA-qAD-pAE+qEA)\|v\rangle$	$(p-q)\langle w\| \quad \cdots \quad ((d-e)A)\|v\rangle$

The last column shows that each block of D's gives rise to two terms, one from each boundary of the block, in which one of the factors of D is replaced by the constant d; these terms have opposite signs and hence cancel. Two similarly cancelling terms arise from each block of E's. Finally, the left and right ends of the product give rise to additional terms in which the original amplitude is multiplied by $(e-d)$ and $(d-e)$, respectively; the cancellation of these terms completes the verification of (A.4) for the weights (A.6).

Appendix B. The invariant measure in the weakly asymmetric limit

In this appendix we verify the picture given in Section 7 of the weakly asymmetric limit of the invariant measure: for $\epsilon = p - q$ very small, the measure is approximately a convolution of a Bernoulli measure having density profile $\rho(y)$ with the density $Q(y)$ of the position of the second class particle in this profile. Here $y = \epsilon n$, so that $\rho(y)$ and $Q(y)$ (see (7.6) and (7.7)) vary on the macroscopic scale .

We begin by describing the key steps in the argument. The probability of occupation numbers $\sigma_1, \sigma_2, \ldots \sigma_m$ at specified sites $n_1 < n_2 < \cdots < n_m$ is given by

$$\langle w|(\sigma_1 D + (1-\sigma_1)E)(D+E)^{n_2-n_1-1}(\sigma_2 D + (1-\sigma_2)E)(D+E)^{n_3-n_2-1}\cdots A\cdots$$
$$\cdots(D+E)^{n_m-n_{m-1}-1}(\sigma_m D + (1-\sigma_m)E)|v\rangle \qquad \text{(B.1)}$$

We want to compute an approximation to (B.1) that is correct in the $\epsilon \to 0$ limit. First, each factor $(D+E)^n$ in the product has matrix elements $(D+E)^n_{ij} = P_n(i-j)$; since $P_n(k)$ is the probability distribution of a biased random walker and is concentrated near $k = n(a-b)$, we approximate $(D+E)^{n-1}_{ij} \simeq (D+E)^n_{ij} \simeq \delta_{i-j,\lfloor n(a-b)\rfloor}$. Next, we approximate each single factor of D, E, or A in the product by a diagonal matrix, obtained from equations (5.9), (5.10), and (5.13) by making first the approximation $L \simeq L^{-1} \simeq I$ and then a simple numerical approximation:

$$D_{ij} \simeq \left(\rho_+ - 2\lambda \frac{x^j}{1+x^j}\right)\delta_{ij} \simeq \rho\left(\frac{\epsilon j}{2\lambda}\right)\delta_{ij}, \tag{B.2}$$

$$E_{ij} \simeq \left(1 - \rho_+ + 2\lambda \frac{x^j}{1+x^j}\right)\delta_{ij} \simeq \left(1 - \rho\left(\frac{\epsilon j}{2\lambda}\right)\right)\delta_{ij}, \tag{B.3}$$

$$A_{ij} \simeq -S_{jj}\delta_{ij} \simeq \frac{\epsilon}{2\lambda} Q\left(\frac{\epsilon j}{2\lambda}\right)\delta_{ij}. \tag{B.4}$$

With these approximations the inner product (B.1) becomes

$$\sum_{j=-\infty}^{\infty} \prod_{k=1}^{m} \left[\sigma_k \rho\left(\frac{\epsilon(j+n_k)}{2\lambda}\right) + (1-\sigma_k)\left(1 - \rho\left(\frac{\epsilon(j+n_k)}{2\lambda}\right)\right)\right] Q\left(\frac{\epsilon j}{2\lambda}\right) \frac{\epsilon}{2\lambda}\,. \tag{B.5}$$

But this is just a Riemann sum for the integral

$$\int_{-\infty}^{\infty} Q(z)\,dz \prod_{k=1}^{m} \left[\sigma_k \rho(y_k + z) + (1-\sigma_k)(1 - \rho(y_k + z)\right]\,, \tag{B.6}$$

where $y_k = \epsilon n_k$. Since the integrand vanishes exponentially fast at $\pm\infty$ there is no difficulty with convergence of the Riemann sums. Moreover, we will sketch below a proof that the sum of the errors made in the approximations leading to (B.5)—that is, the difference between (B.1) and (B.5)—vanishes as $\epsilon \to 0$, uniformly in the choices of n_k and σ_k. Thus we conclude: if the n_k are chosen to depend on ϵ in such a way that $\lim_{\epsilon\to 0} \epsilon n_k = y_k$ then

$$\lim_{\epsilon\to 0} \mathrm{Prob}\{\tau_{n_k} = \sigma_k \mid k = 1, \ldots, m\} = \int_{-\infty}^{\infty} Q(z)\,dz \prod_{k=1}^{m} \left[\sigma_k \rho(y_k + z) + (1-\sigma_k)(1 - \rho(y_k + z)\right]\,. \tag{B.7}$$

Equation (B.7) gives the $\epsilon \to 0$ limiting behavior of the profile, equation (7.8), and the similar behavior (7.10) in the example discussed in Section 7 for the distribution of the occupation numbers of several sites. In general, (B.7) describes the limiting behavior of the distribution of occupation numbers at a finite number of sites with both microscopic and macroscopic spacing, since we do not assume that the positions y_k are distinct.

We now discuss the coarse grained random variables $B_\epsilon(y)$, the empirical densities on an $\epsilon^{-\beta}$ scale, where $0<\beta<1$ (see (7.11)). We want to show that for a typical configuration in the $\epsilon \to 0$ limit these variables lie on the hyperbolic tangent profile $\rho(y)$, after a configuration-dependent translation. Due to this random translation, however, the B_ϵ variables fluctuate even in the $\epsilon \to 0$ limit; to obtain a sharp statement, we show that there are no fluctuations in a function of several of these variables, $g(B) = g(B_\epsilon(y_1), \dots, B_\epsilon(y_K))$, when g is independent of translations of the y variables (see (B.10) below). One example of this technique is (7.12). To show the absence of fluctuations in $g(B)$ we want to show that $\langle |g(B) - \langle g(B)\rangle|\rangle = 0$; the idea then is to use (B.7) to compute expectations of functions of the $B_\epsilon(y)$.

Specifically, suppose that $g(\xi_1, \dots, \xi_K)$ is a function defined for $0 \le \xi_k \le 1$ and that $y_1, \dots y_K$ are any real numbers. We will show that

$$\lim_{\epsilon\to 0}\langle g(B_\epsilon(y_1), \dots, B_\epsilon(y_K))\rangle = \int_{-\infty}^{\infty} g(\rho(y_1+z), \dots, \rho(y_K+z))\, Q(z)\, dz. \tag{B.8}$$

We verify (B.8) first for g a monomial, then a polynomial, then continuous, and finally for g bounded and continuous on $(\rho_-, \rho_+)^K$, the case needed in applications. If g is a monomial, say of degree N, then we may substitute the definition (7.11) of B_ϵ into g and expand, so that $\langle g(B_\epsilon(y_1), \dots, B_\epsilon(y_K))\rangle$ becomes a sum of $\lfloor \epsilon^{-\beta}\rfloor^N$ terms $\epsilon^{-N\beta}\langle \prod_{j=1}^N \tau_{m_j}\rangle$. There are at most $\binom{N}{2}\epsilon^{-(N-1)\beta}$ such terms in which not all the τ_{m_j} are distinct, so that their contribution can be ignored, and from (B.7) the remaining terms give precisely the right hand side of (B.8). Thus (B.8) holds if g is a polynomial and hence, by a uniform approximation argument, if g is continuous on $[0,1]^K$. If we apply the latter result, for $K=1$, to a continuous function $h_\delta(\xi)$ satisfying $0 \le h_\delta(\xi) \le 1$, $h_\delta(\xi) = 1$ for $\rho_- + 2\delta \le \xi \le \rho_+ - 2\delta$, and $h_\delta(\xi) = 0$ for $\xi \le \rho_- + \delta$ and $\xi \ge \rho_+ - \delta$ we have

$$\mathrm{Prob}\{\rho_- + \delta \le B_\epsilon(y) \le \rho_+ - \delta\} \ge \langle h_\delta(B_\epsilon(y))\rangle \underset{\epsilon\to 0}{\longrightarrow} \int_{-\infty}^{\infty} h_\delta(\rho(y+z))\, Q(z)\, dz, \tag{B.9}$$

so that $\mathrm{Prob}\{\rho_- + \delta \le B_\epsilon(y) \le \rho_+ - \delta\} \to 1$ as $\delta, \epsilon \to 0$. Using this result, we may extend (B.8) to any function g which is bounded and is continuous on $(\rho_-, \rho_+)^K$. To do so, we restrict g to $(\rho_- + \delta, \rho_+ - \delta)^K$ and then apply (B.8) to a continuous extension of this restriction satisfying the same bound as g, obtaining (B.8) for g with error which vanishes as $\delta \to 0$.

Now consider a function g as above, bounded and continuous on $(\rho_-, \rho_+)^K$, such that for some $y_1, \dots, y_K$, $g(\rho(y_1), \dots, \rho(y_K))$ is translation invariant: for any z,

$$g(\rho(y_1), \dots, \rho(y_K)) = g(\rho(y_1+z), \dots, \rho(y_K+z)). \tag{B.10}$$

Then

$$\begin{aligned}
&\lim_{\epsilon\to 0} \langle\, |g(B_\epsilon(y_1),\ldots,B_\epsilon(y_K)) - g(\rho(y_1),\ldots,\rho(y_K))\,|\rangle \\
&\qquad = \int_{-\infty}^{\infty} |g(\rho(y_1+z),\ldots,\rho(y_K+z)) - g(\rho(y_1),\ldots,\rho(y_K))|\, Q(z)\, dz \\
&\qquad = 0\,,
\end{aligned} \tag{B.11}$$

so that $\lim_{\epsilon\to 0} g(B_\epsilon(y_1),\ldots,B_\epsilon(y_K)) = g(\rho(y_1),\ldots,\rho(y_K))$ in probability.

From (B.11) we obtain the result described at the end of Section 7: if for any y we define $g(\xi_1,\xi_2) = \xi_2 - \rho(y+\rho^{-1}(\xi_1))$ and then apply (B.11) to $g(B_\epsilon(0),B_\epsilon(y))$, we obtain (7.13). Note that g is bounded but is not continuous at $\xi_1=\xi_2=\rho_-$ or $\xi_1=\xi_2=\rho_+$.

We finally sketch the argument that the errors made in the approximations leading to (B.5) are uniformly small. Control of errors introduced by the approximation $(D+E)^n_{ij} \simeq \delta_{i-j,\lfloor n(a-b)\rfloor}$ is straightforward, and details will be omitted. To discuss (B.2)–(B.4), we simplify the notation by supposing that $n_1 > 0$. Let us write $D^{(0)}$ for the diagonal approximation to D introduced in (B.2), and $D^{(u)} = uD + (1-u)D^{(0)}$ for $0 \le u \le 1$, with similar notation for E and A. We must then estimate

$$\begin{aligned}
&\int_0^1 \frac{d}{du}\langle w|A^{(u)}(D+E)^{n_1-1}(\sigma_1 D^{(u)} + (1-\sigma_1)E^{(u)})\ \cdots\ |v\rangle\, du \\
&\qquad = \sum_{j=1}^{m}\int_0^1 \langle w|A^{(u)}\ \cdots\ (\sigma_j[D-D^{(0)}] + (1-\sigma_j)[E-E^{(0)}])\ \cdots\ |v\rangle\, du \\
&\qquad\qquad + \int_0^1 \langle w|[A-A^{(0)}](D+E)^{n_1-1}\ \cdots\ |v\rangle\, du\ .
\end{aligned} \tag{B.12}$$

On the right hand side of (B.12) we regard $\langle w|$ and $|v\rangle$ as elements (of norm 1) of $\ell^\infty = \ell^\infty(\mathbb{Z})$, and note that D, E, $D^{(0)}$, $E^{(0)}$, and $D+E$ are bounded operators on ℓ^∞, of norm at most 1, and that A and $A^{(0)}$ are bounded operators from ℓ^∞ to $\ell^1(\mathbb{Z})$. Now consider one term from the sum over j in (B.12), for which the matrix product will contain a factor of either $D-D^{(0)}$ or $E-E^{(0)}$. Each of these may be written as the sum of two terms, corresponding to the two approximations made in (B.2) or (B.3). The first of these terms contains a factor $L-I$, which we carry to the right until it reaches, and annhilates, $|v\rangle$; $L-I$ commutes with $D+E$ and its commutator with $D^{(u)}$ or $E^{(u)}$ has ℓ^∞ operator norm which is $O(\epsilon)$. The second error term also has ℓ^∞ operator norm which is $O(\epsilon)$. The contribution to (B.12) containing $A-A^{(u)}$ is treated similarly, and we derive an overall bound for (B.12) which is a constant multiple of $m^2\epsilon$.

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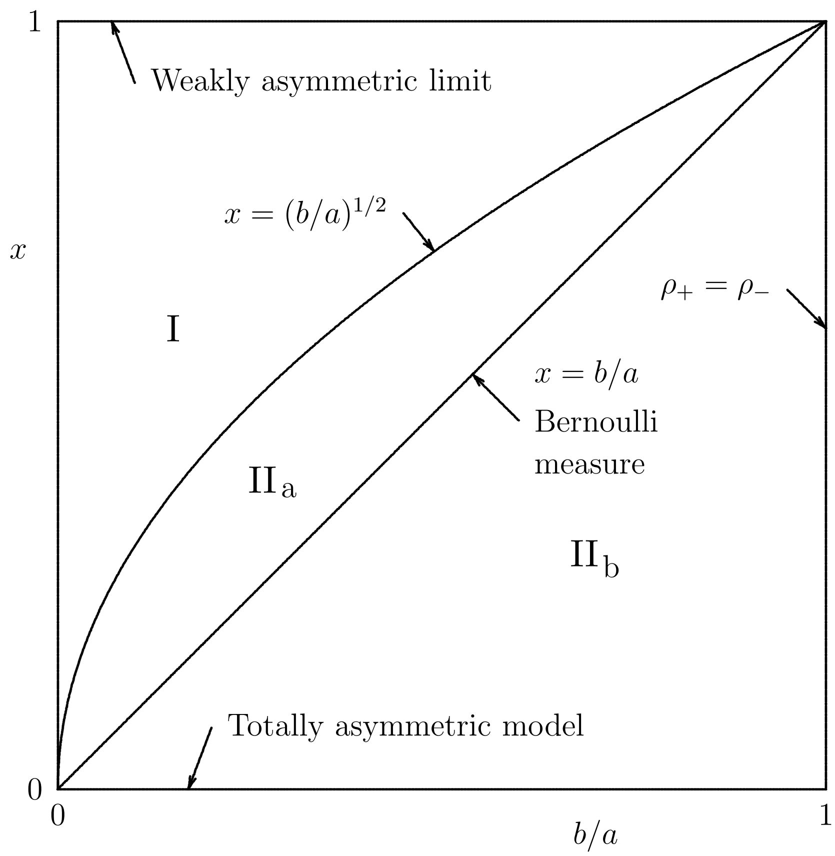
FIGURE CAPTIONS

Figure 1. A phase diagram for the asymptotics of the shock in the ASEP. For $n >> 1$, $\langle\tau_n\rangle \simeq \rho_+ + Cn^\gamma \exp -\kappa n$, with $\kappa = -\log(1 - a - b + ax + b/x)$ and $\gamma = 1$ in region I and $\kappa = -\log(1 - (\sqrt{a} - \sqrt{b})^2)$ and $\gamma = -3/2$ in region II, and with $C < 0$ in regions I and $\mathrm{II_a}$ and $C > 0$ in region $\mathrm{II_b}$.

Figure 2. Profile $\langle\tau_n\rangle$ for $\rho_- = 0.3$, $\rho_+ = 0.6$: $x = 0.0$ ($\bullet$); $x = x^* = 0.2857$ ($\times$); $x = 0.6$ ($+$). The dashed line is at height $(\rho_- + \rho_+)/2$.



1
Weakly asymmetric limit
$x = (b/a)^{1/2}$
x
$\rho_+ = \rho_-$
I
$x = b/a$
Bernoulli
measure
II_a
II_b
Totally asymmetric model
0
0
b/a
1

Figure 1

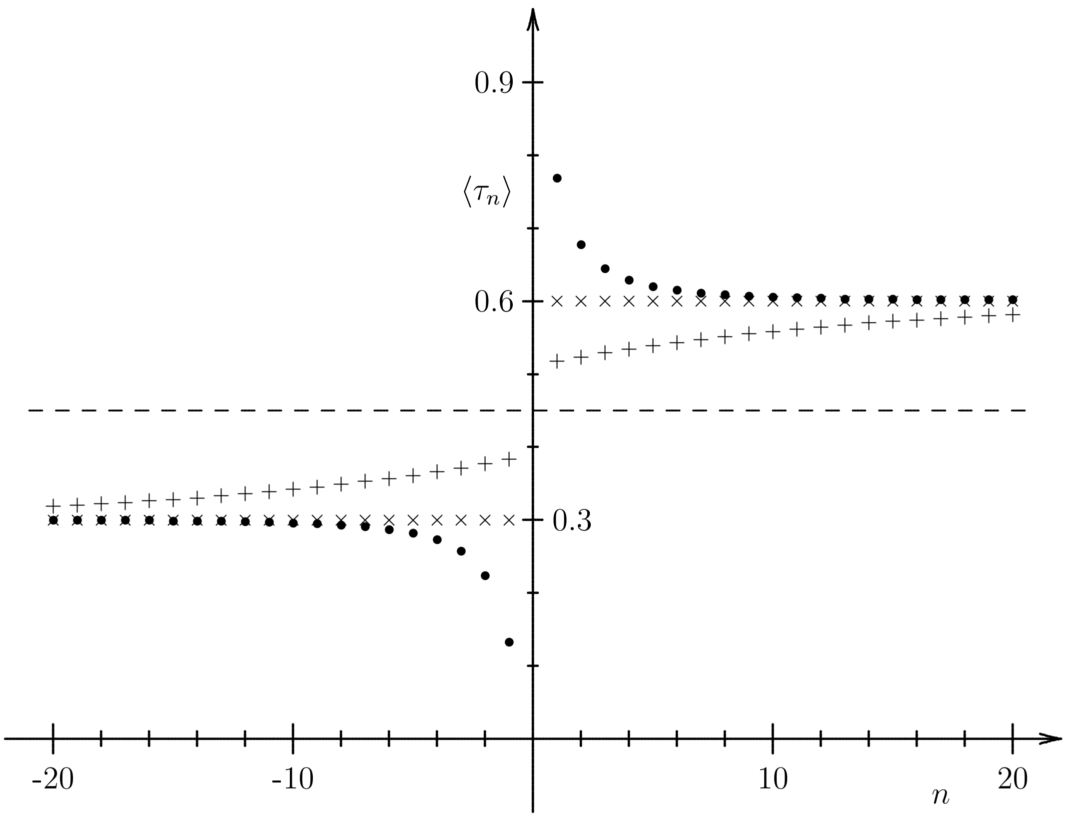


Figure 2